

# Quantum Holonomies for Quantum Computing

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Holonomic Quantum Computation (HQC) is an all-geometrical approach to quantum information processing. In the HQC strategy information is encoded in degenerate eigen-spaces of a parametric family of Hamiltonians. The computational network of unitary quantum gates is realized by driving adiabatically the Hamiltonian parameters along loops in a control manifold. By properly designing such loops the non-trivial curvature of the underlying bundle geometry gives rise to unitary transformations i.e., holonomies that implement the desired unitary transformations. Conditions necessary for universal QC are stated in terms of the curvature associated to the non-abelian gauge potential (connection) over the control manifold. In view of their geometrical nature the holonomic gates are robust against several kind of perturbations and imperfections. This fact along with the adiabatic fashion in which gates are performed makes in principle HQC an appealing way towards universal fault-tolerant QC.

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## I. INTRODUCTION

It is by-now a generally accepted fact that the laws of quantum theory provide in principle a radically novel, and more powerful way to process information with respect to any conceivable classically operating device [1]. In the last few years a big deal of activity has been devoted to devise and to implement schemes for taking actual advantage from this extra quantum power. In particular in Quantum Computation the states of a quantum system  $S$  are used for encoding information in such a way that the final state, obtained by the unitary time-evolution of  $S$ , encodes the solution of a given computational problem. A system  $S$  with state-space  $\mathcal{H}$  (the Quantum Computer) supports universal QC if any unitary transformation  $U \in \mathcal{U}(\mathcal{H})$  can be approximated with arbitrarily high accuracy by a sequence (the *network*) of simple unitaries (the *gates*) that the experimenter is supposed to be able to implement. The case in which  $S$  is a multi-partite system is the most important one as it allows for *entanglement*, a unique quantum feature that is generally believed to be one of the crucial elements from which polynomial or exponential speed-up occurs [2].

In the above picture of QC the realization of the quantum network is achieved at the physical level by turning on and off external fields coupled to  $S$  as well as *local* interactions among the subsystems of  $S$ . In other words the experimenter “owns” a basic set of time-dependent Hamiltonians that she/he activates at will to perform the suitable sequences of quantum logic gates. At variance with such a standard *dynamical* view of QC more recently several authors considered *geometrical* and *topo-*

*logical* approaches [3–6]. The peculiarity of these proposals is somehow striking: over the manifold  $\mathcal{C}$  of quantum codewords one can have a trivial Hamiltonian e.g.,  $H|_C = 0$ , nevertheless, obtaining a non-trivial quantum evolution due to the existence of an underlying geometrical/topological *global* structure. The quantum gates - or a part of them - are then realized in terms of operations having a purely geometrical/topological nature. Besides being conceptually intriguing on their own, these schemes have some built-in fault-tolerant features. This latter attractive characteristic stems from the fact that some topological as well as geometrical quantities are inherently stable against local perturbations. This in turn allows for Quantum Information processing inherently stable against special classes of computational errors.

In this paper we shall give a detailed account of *Holonomic Quantum Computation* (HQC) introduced in Ref. [7] and further developed in Refs. [8], [9], [10] and [11]. In this novel gauge-theoretic framework one is supposed to be able to control a set of parameters  $\lambda \in \mathcal{M}$ , on which depends an iso-degenerate family  $\mathcal{F}$  of quantum Hamiltonians  $\{H(\lambda)\}$ . Information is encoded in an  $n$ -dimensional eigen-space  $\mathcal{C}$  of a distinguished  $H(\lambda_0) \in \mathcal{F}$ . Universal QC [13] over  $\mathcal{C}$  can be then obtained by adiabatically driving the control parameters along suitable loops  $\gamma$  rooted at  $\lambda_0$ . The key physical ingredient is provided by the appearance in such quantum evolutions of non-Abelian geometric contributions [14]  $U_\gamma \in U(n)$  ( $n > 1$ ) given by *holonomies* associated with a  $u(n)$ -valued gauge potential  $A$  [12,15]. In other words quantum computation in the HQC approach is nothing but the parallel transport of states in  $\mathcal{C}$  realized by the *connection*  $A$ .

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Therefore the computational power in the HQC approach relies on the non-triviality of the geometry of the bundle of eigen-spaces of  $\mathcal{F}$  over the manifold of control parameters,  $\mathcal{M}$ . It must be noticed that also in Refs. [5,6], even though the evolutions are not adiabatic, holonomies play an important role. Nevertheless the connections involved in those approaches are *Abelian* giving rise to  $U(1)$ -valued holonomies e.g., Berry phases [16]. It is then clear that in order to achieve universality these geometric gates must be supplemented by standard dynamical operations. On the other hand in the HQC approach *the whole quantum network is built by means of holonomies*. In this sense HQC is *fully* geometrical. It is worth observing that the computational subspace  $\mathcal{C}$  can be thought of as the lowest-energy manifold of a highly symmetric quantum system; from this point of view HQC is a kind of *ground-state* computation. This last remark points out the potential existence of a fault-tolerant [17] feature of HQC due to energy gaps and even spontaneous relaxation mechanisms. Further fault-tolerant characteristics, of HQC models we considered, are related to the fact that the holonomies  $U_\gamma$  realizing quantum computations turn out to depend just on the *areas* of the surfaces that the generating loops  $\gamma$  span on certain two-dimensional sub-manifolds. When this area is given one can consider even very large i.e., “far” from the identity deformations of  $\gamma$ , but as long as they are area-preserving no errors are induced. Moreover as far as the adiabaticity condition holds  $U_\gamma$  does not depend on the rate at which the control loops are driven. Hence, even with respect the issue of timing, HQC is robust.

The paper is organized as follows. In Section II the general theory of holonomic quantum evolution is presented as well as its application to quantum information processing. A new pedagogical proof of the adiabatic theorem is given and the properties of the holonomies are analyzed, facilitating their application to quantum computing. In Section III the  $\mathbf{CP}^n$  model is presented as a theoretical holonomic arena allowing for the analytic evaluation of the connection  $A$ , its field strength  $F$  and a complete set of calculated holonomies. The mathematical steps which enable such a calculation are given in detail providing an analytical method for calculating the holonomies (Wilson loops) much used in many areas of theoretical physics. In Section IV a physical model is presented based on quantum optics. Known optical devices as displacers, squeezers and interferometers are employed as control devices performing coherent evolution of the laser photon states. The laser beams propagate in a Kerr medium which provides the desired degeneracy. Further similar applications to harmonic oscillator setups are discussed, which are experimentally more viable. In the Appendix a more mathematical approach to the Holonomic evolutions is presented.

## II. GENERAL THEORY

In this section the general theoretical framework of HQC is reviewed. While the exposition relies partly on Refs. [7,8] some proofs have been added which clarify the physical concept of holonomic evolutions making this subject more approachable to the quantum information community.

### A. Quantum Evolutions

Let us suppose that we have at disposal a family  $\mathcal{F}$  of Hamiltonians that we can turn on and off in order to let an  $N$ -dimensional quantum system to evolve in a controllable way. Formally, we assume  $\mathcal{F} := \{H(\lambda)\}_{\lambda \in \mathcal{M}}$  to be a continuous family of Hermitian operators over the state-space  $\mathcal{H} \cong \mathbb{C}^N$ . The parameters  $\lambda$  on which the elements of  $\mathcal{F}$  depend will be referred to as *control* parameters and their manifold  $\mathcal{M}$  as the *control manifold*, thought to be embedded in  $\mathbb{R}^{N^2}$ . Indeed, one has

$$H(\lambda) = i \sum_{a=1}^{N^2} \Phi_a(\lambda) T_a \in u(N)$$

where the  $T_a$ 's constitute a basis of the  $N^2$ -dimensional Lie-algebra  $u(N)$  of anti-hermitian matrices, and  $\Phi: \mathcal{M} \mapsto u(N)$  is a smooth mapping that associates to any  $\lambda$  in the control manifold a vector in  $u(N)$  with  $T$ -components  $(\Phi_1(\lambda), \dots, \Phi_{N^2}(\lambda))$ .

The evolution of the quantum system is thought of as actively driven by the parameters  $\lambda$ , over which the experimenter is assumed to have direct access and controllability. Suppose we are able to drive by a *dynamical control process* the parameter configuration  $\lambda \in \mathcal{M}$  through a path  $\gamma: [0, T] \rightarrow \mathcal{M}$ . Hence, a one-parameter i.e., time-dependent family

$$\mathcal{F}_\gamma := \{H(t) := H[\Phi \circ \gamma(t)] : t \in [0, T]\} \subset \mathcal{F} \quad (1)$$

is defined. Notice that even the converse is true: any smooth family  $\{H(t)\}_{t \in [0, T]}$  defines a path in  $\mathcal{M} = \mathbb{R}^{N^2}$ . The quantum evolution associated to the time-dependent family (1) is described by the time-dependent Schrödinger equation  $i \partial_t |\psi(t)\rangle = H(t) |\psi(t)\rangle$  and hence it has the operator form

$$U_\gamma := \mathbf{T} \exp\{-i \int_0^T dt H(t)\} \in U(N) \quad (2)$$

where  $\mathbf{T}$  denotes chronological ordering. The time-dependent quantum evolution (2), for a given map  $\Phi$ , depends in general on the path  $\gamma$  and *not* just on the curve  $\gamma([0, T])$  i.e., the image of  $\gamma$  in the control manifold. In other words the unitary transformation (2) contains a *dynamical* as well as a *geometrical* contribution,

the former depends even on the *rate* at which  $\gamma([0, T])$  is traveled along whereas the latter depends merely on the geometrical characteristics of the curve.

From the physical point of view the parameters  $\lambda$  represent in general external fields and, for multi-partite systems, couplings among the various subsystems. To illustrate this point let us consider  $\mathcal{H} := (\mathbb{C}^2)^{\otimes N} \cong \mathbb{C}^{2^N}$  i.e., a  $N$ -qubit system. Then a basis for  $u(2^N)$  is provided by the tensor products  $T_\alpha := \otimes_{i=1}^N \hat{\sigma}_{\alpha_i}$  where  $\alpha: \{1, \dots, N\} \mapsto \{0, 1, 2, 3\}$  and  $\hat{\sigma}_0 := \mathbb{1}$ ,  $\hat{\sigma}_1 := \sigma_x$ ,  $\hat{\sigma}_2 := \sigma_y$ ,  $\hat{\sigma}_3 := \sigma_z$  are the Pauli matrices. It is then clear that any  $\alpha$  which takes a non-zero value more than once e.g.,  $\alpha_i \alpha_j \neq 0$  describes a non-trivial interaction which generates entanglement between the qubits  $i$  and  $j$ . Therefore the ability to manipulate the weight of the contribution of  $T_\alpha$ 's in the decomposition of  $H(\lambda)$ , amounts to the capacity of dynamically controlling many-body couplings. This goal is, of course, even conceptually more difficult to achieve than the control of the real external fields, namely the interaction associated to single subsystem generators  $T_\alpha$ . Finally, we stress that there is still another possibility; the control parameters  $\lambda$  could represent on *their own* quantum-degrees of freedom e.g., nuclear coordinates in the adiabatic approximation for molecular systems, treated in some quasi-classical fashion. This situation arises when one performs an adiabatic decoupling between “fast” and “slow” degrees of freedom, getting for the former a Hamiltonian that depends parametrically on the latter [16]. In this case the control manifold  $\mathcal{M}$  is nothing but the classical configuration manifold associated with a quantum system.

Within this framework the requirements for implementing universal QC [13] can be expressed in terms of the availability of paths. Universality is the experimental capability of driving the control parameters along a minimal set  $\{\gamma_i\}_{i=1}^g$  of paths which generate the basic unitary transformations  $U_{\gamma_i}$ 's, i.e. the gates. By sufficiency of this set we mean the ability to approximate any  $U \in U(N)$  with arbitrarily high accuracy by means of path sequences.

## B. Holonomies

Now we recall some basic facts about quantum holonomies. A more mathematical approach can be found in Appendix A, where some by-now standard material has been collected aiming to make the paper as much as possible self-contained.

The non-Abelian holonomies are a natural generalization of the Abelian Berry phases. We first assume that  $\mathcal{F}$  is an *iso-degenerate* Hamiltonian family i.e., all the elements of  $\mathcal{F}$  have the same degeneracy structure. This means that a generic Hamiltonian of  $\mathcal{F}$  can be written as  $H(\lambda) = \sum_{l=1}^R \varepsilon_l(\lambda) \Pi_l(\lambda)$  where  $\Pi_l(\lambda)$  denotes the pro-

jector over the eigen-space  $\mathcal{H}_l(\lambda) := \text{span}\{|\psi_l^\alpha(\lambda)\rangle\}_{\alpha=1}^{n_l}$ , with eigenvalues  $\varepsilon_l(\lambda)$ , whose dimension  $n_l$  is independent on the control parameter  $\lambda$ . In order to preserve the  $R$  degeneracies  $n_l$  we also assume that over  $\mathcal{M}$  there is no level-crossing i.e.,  $l \neq l' \Rightarrow \varepsilon_l(\lambda) \neq \varepsilon_{l'}(\lambda), \forall \lambda \in \mathcal{M}$ . In addition, we shall restrict to *loops*  $\gamma$  in the control manifold i.e., maps  $\gamma: [0, T] \mapsto \mathcal{M}$  such that  $\gamma(0) = \gamma(T)$ . These conditions in the dynamics of the system and in the control manipulations will facilitate the generation of holonomic unitaries.

Let us state the main result [14] on which the HQC relies. Consider a system with the above characteristics. When its control parameters are driven adiabatically i.e., slow with respect to any time-scale associated to the system dynamics, along a loop  $\gamma$  in  $\mathcal{M}$  any initially prepared state  $|\psi_{in}\rangle \in \mathcal{H}$  will be mapped after the period  $T$  onto the state

$$|\psi_{out}\rangle = U_\gamma |\psi_{in}\rangle, \quad U_\gamma = \oplus_{l=1}^R e^{i\phi_l} \Gamma_{A_l}(\gamma), \quad (3)$$

where,  $\phi_l := \int_0^T d\tau \varepsilon_l(\lambda_\tau)$ , is the dynamical phase whereas the matrices  $\Gamma_{A_l}(\gamma)$ 's represent the geometrical contributions. They are unitary mappings of  $\mathcal{H}_l$  onto itself and they can be expressed by the following path ordered integrals

$$\Gamma_{A_l}(\gamma) := \mathbf{P} \exp \oint_\gamma A_l \in U(n_l), \quad l = 1, \dots, R. \quad (4)$$

These are the *holonomies* associated with the loop  $\gamma$ , and the *adiabatic connection forms*  $A_l$ . The latter have an explicit matrix form given by  $A_l = \Pi_l(\lambda) d\Pi_l(\lambda) = \sum_\mu A_{l,\mu} d\lambda_\mu$ , where [16] analytically

$$(A_{l,\mu})^{\alpha\beta} := \langle \psi_l^\alpha(\lambda) | \partial/\partial\lambda^\mu | \psi_l^\beta(\lambda) \rangle \quad (5)$$

with  $(\lambda_\mu)_{\mu=1}^d$  the local coordinates on  $\mathcal{M}$ . The connection forms  $A_l$ 's are nothing but the non-Abelian gauge potentials enabling the parallel transport [15] over  $\mathcal{M}$  of vectors of the fiber  $\mathcal{H}_l$ . Result (5) is the non-Abelian generalization of the Berry phase connection presented first by Wilczek and Zee (1984) (see Appendix). Due to the decomposition of the evolution operator in (3) into distinct evolutions for each eigen-space  $\mathcal{H}_l$ , we are able to restrict our study to a given degenerate eigen-space with fixed  $l$ .

We shall present first an intuitive proof for deriving (4) and (5), aiming in clarifying the gauge structure interpretation of this adiabatic evolution and in providing a more physical insight. Without loss of generality we shall assume the family  $\mathcal{F}$  to be *iso-spectral*. This implies that for any  $\lambda \in \mathcal{M}$  it exists a unitary transformation  $\mathcal{U}(\lambda)$  such that  $H(\lambda) = \mathcal{U}(\lambda) H_0 \mathcal{U}(\lambda)^\dagger$ , where  $H_0 := H(\lambda_0)$ . Upon dividing the time interval  $[0, T]$  into  $N$  equal segments  $\Delta t$ , for  $\mathcal{U}_i = \mathcal{U}(\gamma(\lambda(t_i)))$  one obtains the evolution operator in the form

$$\begin{aligned}
U_\gamma &= \mathbf{T} e^{-i \int_0^T \mathcal{U}(\lambda) H_0 \mathcal{U}^\dagger(\lambda) dt} = \mathbf{T} \lim_{N \rightarrow \infty} e^{-i \sum_{i=1}^N \mathcal{U}_i H_0 \mathcal{U}_i^\dagger \Delta t} \\
&= \mathbf{T} \lim_{N \rightarrow \infty} \prod_{i=1}^N \mathcal{U}_i e^{-i H_0 \Delta t} \mathcal{U}_i^\dagger \quad (6)
\end{aligned}$$

The third equality holds due to the smallness of the interval  $\Delta t$  in the limit of large  $N$ . The product  $\mathcal{U}_i^\dagger \mathcal{U}_{i+1}$  of two successive unitaries, gives rise to an infinitesimal rotation of the form  $\mathcal{U}_i^\dagger \mathcal{U}_{i+1} \approx \mathbb{1} + \vec{A}_i \cdot \Delta \vec{\lambda}_i$ , where  $(A_i)_\mu \equiv \mathcal{U}_i^\dagger \frac{\Delta \mathcal{U}_i}{\Delta (\lambda_i)_\mu}$ . The connection  $A$  has at time  $t_i$  the components  $(A_i)_\mu$  with  $\mu = 1, \dots, d$ . Hence the evolution operator (6) becomes

$$\mathbf{T} \lim_{N \rightarrow \infty} \mathcal{U}_N \left( \mathbb{1} - i H_0 N \cdot \Delta t + \sum_{i=1}^{N-1} \vec{A}_i \cdot \Delta \vec{\lambda}_i \right) \mathcal{U}_1^\dagger. \quad (7)$$

For the case of a closed path the initial and the final transformations  $\mathcal{U}_1$  and  $\mathcal{U}_N$  are identical as they correspond to the same point of the control parameter manifold. With a reparametrization they may be taken to be equal to the identity transformation. Now we consider an initial state  $|\psi_{in}\rangle$  belonging to an eigen-space  $\mathcal{H}_0$  with associated eigenvalue e.g.,  $\varepsilon_0 = 0$ . Due to the time ordering symbol the actions on the state  $|\psi(t)\rangle$  of the Hamiltonian and of the connection  $A$  are alternated, hence in general we cannot separate them into two exponentials. On the other hand, if we demand adiabaticity, namely very slow exchange of energy during the process, this will keep the state within  $\mathcal{H}_0$ , then at each time  $t_i$  the state  $|\psi(t_i)\rangle$  will remain in the  $\varepsilon_0 = 0$  energy level. This allows to factor out in (7) the action of  $H$ , thus obtaining

$$U_\gamma = \mathbf{T} \lim_{N \rightarrow \infty} \left( \mathbb{1} + \sum_{i=1}^{N-1} \vec{A}_i \cdot \Delta \vec{\lambda}_i \right) = \mathbf{P} \exp \oint_\gamma A,$$

where  $A$  is projected into the subspace  $\mathcal{H}_0$ . Notice that we replaced the time ordering with the path ordering  $\mathbf{P}$  as the parameter of the integration at the last expression is the position on the loop  $\gamma$ . In this proof of the non-Abelian geometrical evolution it is clear how the holonomy appears and which physical conditions enable its formation. In the same way we could have considered in addition to the equivalent transformations of the Hamiltonian a multiplicative function  $\varepsilon_0(t)$  varying the energy eigenvalue. The results would be unaltered apart from the insertion of a dynamical phase.

Let us now view some of the properties the holonomies have in terms of gauge reparametrization of the connection and loop composition rules. In our context a local gauge transformation is the unitary transformation  $\mathcal{U}(\lambda) \mapsto \mathcal{U}(\lambda)g(\lambda)$ , which does not change the Hamiltonian operator  $H_0$ . Its action merely reparametrizes the variables of the control manifold. Taking into account the properties  $gH_0 = H_0g$  and  $g\Pi = \Pi g$  we are able to obtain the transformation of the connection as

$A \mapsto g^\dagger A g + g^\dagger dg$ , ( $g \in U(n)$ ). It immediately follows that the holonomy transforms as  $\Gamma_A \mapsto g^\dagger \Gamma_A g$ . Notice that in the new coordinates the state vectors  $|\psi\rangle$  i.e., the sections, become  $g^\dagger |\psi\rangle$ . This property makes it clear that the holonomy transformation has an intrinsic i.e., coordinate-free, meaning. Furthermore, the holonomy has the following property in terms of the loops. We define (setting  $T = 1$ ) the loop space at a given point  $\lambda_0 \in \mathcal{M}$  as

$$L_{\lambda_0} := \{\gamma: [0, 1] \mapsto \mathcal{M} / \gamma(0) = \gamma(1) = \lambda_0\}$$

over a point  $\lambda_0 \in \mathcal{M}$ . Let us stress that, as far as the manifold  $\mathcal{M}$  is connected, the distinguished point  $\lambda_0$  does not play any role. In this space we introduce a composition law for loops

$$(\gamma_2 \cdot \gamma_1)(t) = \theta\left(\frac{1}{2} - t\right) \gamma_1(2t) + \theta\left(t - \frac{1}{2}\right) \gamma_2(2t - 1) \quad (8)$$

and a unity element  $\gamma_0(t) \equiv \lambda_0$ ,  $t \in [0, 1]$  moreover with  $\gamma^{-1}$  we shall denote the loop  $t \mapsto \gamma(1 - t)$ .

The holonomy can be considered as a map  $\Gamma_A: L_{\lambda_0} \mapsto U(n)$ , whose basic properties can be easily derived from eq. (4):

- i)  $\Gamma_A(\gamma_2 \cdot \gamma_1) = \Gamma_A(\gamma_2) \Gamma_A(\gamma_1)$ ; by composing loops in  $\mathcal{M}$  one obtains a unitary evolution that is the product of the evolutions associated with the individual loops,
- ii)  $\Gamma_A(\gamma_0) = \mathbb{1}$ ; staying at rest in the parameter space corresponds to no evolution at all,
- iii)  $\Gamma_A(\gamma^{-1}) = \Gamma_A^{-1}(\gamma)$ ; in order to get the inverse holonomy one has to traverse the path  $\gamma$  with reversed orientation,
- iv)  $\Gamma_A(\gamma \circ \varphi) = \Gamma_A(\gamma)$ , where  $\varphi$  is any diffeomorphism of  $[0, 1]$ ; as long as adiabaticity holds the holonomy does not depend on the speed at which the path is traveled but just on the path geometry.

From the properties listed above it is easy to show that the set  $\text{Hol}(A) := \Gamma_A(L_{\lambda_0})$  is a *subgroup* of  $U(n)$ . Such a subgroup is known as the *holonomy group* of the connection  $A$ . When the holonomy group coincides with the whole  $U(n)$  then the connection  $A$  is called *irreducible*. The notion of irreducibility plays a crucial role in HQC in that it corresponds to the computational notion of *universality* [13]. In order to evaluate if this condition is fulfilled by a given connection it is useful to consider the *curvature* 2-form  $F = \sum_{\mu\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu$  associated with the 1-form connection  $A$  whose components

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (9)$$

The relation of the curvature with irreducibility is given by the following statement [15]: *the linear span of the  $F_{\mu\nu}$ 's is the Lie algebra of the holonomy group*. It follows in particular that when the  $F_{\mu\nu}$ 's span the whole  $\mathfrak{u}(n)$  the connection is irreducible.

### C. Holonomic Quantum Computation

The unitary holonomies (4) are the main ingredient of our approach to QC. From now on we shall consider a given subspace  $\mathcal{H}_l$  (accordingly the label  $l$  will be dropped). Such a subspace, denoted by  $\mathcal{C}$ , will represent our quantum *code*, whose elements will be the quantum information encoding codewords. The crucial remark [7] is that *when the connection is irreducible, for any chosen unitary transformation  $U$  over the code there exists a path  $\gamma$  in  $\mathcal{M}$  such that  $\|\Gamma_A(\gamma) - U\| \leq \epsilon$ , with  $\epsilon$  arbitrarily small.* This means that *any computation on the code  $\mathcal{C}$  can be realized by adiabatically driving the control parameter configuration  $\lambda$  along a suitable closed path  $\gamma$ .*

In particular we aim to constructing specific logical gates by moving along their corresponding loops. Initially, the degenerate states are prepared in to a “ground” state, interpreting the  $|0\dots 0\rangle$  state of  $m$  qubits. The statement of irreducibility of the connection  $A$  relates a particular unitary  $U$  with the loop  $\gamma_U$  over which the connection is integrated to give  $\Gamma_A(\gamma_U) = U$ . Hence, there are loops in the control space such that the associated holonomies give, for example, a one qubit Hadamard gate or a two qubit “controlled-not” gate.

Let us emphasize the fact that one can perform *universal* QC by only using quantum holonomies is remarkable. Indeed this kind of quantum evolutions is quite special, yet it contains in a sense the full computational power. On the other hand one has to pay the price given by the restriction of the computational space from  $\mathcal{H}$  to its subspace  $\mathcal{C}$ . Notice that, for the irreducibility property to hold, a necessary condition is clearly given by  $d(d-1)/2 \geq n^2$  where  $d := \dim \mathcal{M}$ . In particular this implies that for an exponentially large code  $\mathcal{C}$  one has to be able to manipulate an exponentially large number of control parameters.

Moreover like in any other scheme for QC, once the computation is completed a final state measurement is performed. To this aim it could be useful to lift the energy degeneracy in order to be able address energetically the different codewords [8]. This can be done by switching on an external perturbation in a coherent fashion.

We conclude this section by discussing the *computational complexity* issue. The computational subspace  $\mathcal{C}$  does not have in general a tensor product structure. This means that it cannot be viewed in a natural way as the state-space of a multi-partite system for which the notion of quantum entanglement makes sense. The latter, on the other hand, is known to be one of the crucial ingredients that provides to QC its additional power with respect to classical computation. It follows that, from this point of view, the scheme for HQC described so far is potentially incomplete. Indeed – as it will be illustrated later by explicit examples – if  $N = \dim \mathcal{C} = 2^k$  i.e., we encode in  $\mathcal{C}$   $k$  qubits, then for obtaining with a multi-partite struc-

ture a universal set of gates one needs  $O(N)$  elementary holonomic loops. Thus in general one has an *exponential* slow-down in computational complexity.

In Ref. [8] we argued how one can in principle overcome such a drawback by focusing on a class of HQC models with a multi-partite structure given from the very beginning. The basic idea is simple: one considers an holonomic family  $\mathcal{F}$  associated to a genuine multi-partite quantum system such that local (one- and two-qubit) gates can be performed by holonomies. Then from standard universality results of QC [13] stems that efficient quantum computations can be performed. An explicit example of the above strategy is formalized as follows [8].

Let us consider  $N$  *qu-trits*. The state space is then given by  $\mathcal{H}_j \cong \mathbb{C}^3 = \text{span}\{|\alpha\rangle_j / \alpha = 0, 1, 2\}$ . The holonomic (iso-spectral) family has the built-in local structure  $\mathcal{F} = \{H_{ij}(\lambda_{ij})\}$  where the local Hamiltonians  $H_{ij}$  have a non trivial actions only on the  $i$ -th and  $j$ -th factors of  $\mathcal{H}$ . Moreover,  $H_{ij}$  admits a four-dimensional degenerate eigen-space  $\mathcal{C}_{ij} := \text{span}\{|\alpha\rangle_i \otimes |\beta\rangle_j / \alpha, \beta = 0, 1\} \subset \mathcal{H}_i \otimes \mathcal{H}_j \cong \mathbb{C}^9$ . If the  $H_{ij}$ ’s allow for universal HQC over  $\mathcal{C}_{ij}$  then universal QC can be *efficiently* implemented over

$$\mathcal{C} := \text{span}\{\otimes_{i=1}^N |\alpha_i\rangle_i / \alpha_i = 0, 1\} \cong (\mathbb{C}^2)^{\otimes N}.$$

### III. THE $\mathbf{CP}^N$ HOLONOMIC CONSTRUCTION

In this section we shall consider a theoretical model where the holonomic ideas can be materialized. The  $\mathbf{CP}^n$  model will be considered for which the Hamiltonian due to its degeneracy has such a symmetric structure as to allow the control manifold  $\mathcal{M}$  to be the  $n$ -dimensional complex projective space  $\mathbf{CP}^n$ . For quantum computation we are interested in finding the particular loops which generate various holonomic gates and eventually constructing a complete set of universal gates.

The path ordering prescription given in (A7) makes hard the explicit analytical evaluation of the holonomies. In order to tackle this problem we employ two procedures. On the one hand we study loops restricted onto particular two dimensional sub-manifolds having easily calculated holonomies. This geometric restriction overcomes the difficulties connected with path ordering, allowing the evaluation of a complete set of basic holonomies. On the other hand it is possible to compose a generic unitary operator with combinations of elements within this set. Eventually, by the loop composition properties of the holonomies it is possible to find its corresponding composed loop. Even if the techniques presently known do not give the possibility to calculate the holonomy of the most general loop, we shall obtain families of loops and their corresponding holonomies from which any desirable group element may be constructed.

### A. The $\mathbf{CP}^n$ Model

Consider the degenerate Hamiltonian  $H_0 = \varepsilon|n+1\rangle\langle n+1|$  acting on the state-space  $\mathcal{H} \cong \mathbb{C}^{n+1} = \text{span}\{|\alpha\rangle\}_{\alpha=1}^{n+1}$ . We shall take as the family  $\mathcal{F}$  the whole orbit  $\mathcal{O}(H_0) := \{\mathcal{U}H_0\mathcal{U}^\dagger / \mathcal{U} \in U(n+1)\}$  of  $H_0$  under the adjoint action of the unitary group  $U(n+1)$ . This orbit is isomorphic to the  $n$ -dimensional complex projective space

$$\mathcal{O}(H_0) \cong \frac{U(n+1)}{U(n) \times U(1)} \cong \frac{SU(n+1)}{U(n)} \cong \mathbf{CP}^n.$$

Each point,  $\mathbf{z}$ , of the  $\mathbf{CP}^n$  manifold corresponds to a unitary matrix  $\mathcal{U}(\mathbf{z}) = U_1(z_1)U_2(z_2)\dots U_n(z_n)$ , where  $U_\alpha(z_\alpha) = \exp[G_\alpha(z_\alpha)]$  with  $G_\alpha(z_\alpha) = z_\alpha|\alpha\rangle\langle n+1| - \bar{z}_\alpha|n+1\rangle\langle\alpha|$  and  $z_\alpha = \theta_\alpha e^{i\phi_\alpha}$ , for  $\alpha = 1, \dots, n$ . We shall assume in the following the set  $(\boldsymbol{\theta}, \boldsymbol{\phi})$  as real coordinates for  $\mathbf{CP}^n$ . The eigen-states of the rotated Hamiltonians are

$$|\alpha(\boldsymbol{\theta}, \boldsymbol{\phi})\rangle := \mathcal{U}(\boldsymbol{\theta}, \boldsymbol{\phi})|\alpha\rangle = \cos\theta_\alpha|\alpha\rangle -$$

$$\exp(-i\phi_\alpha)\sin\theta_\alpha \sum_{j>\alpha}^{n+1} \exp(i\phi_j)\sin\theta_j \prod_{j>\gamma>\alpha} \cos\theta_\gamma|j\rangle \quad (10)$$

and

$$|n+1(\boldsymbol{\theta}, \boldsymbol{\phi})\rangle := \mathcal{U}(\boldsymbol{\theta}, \boldsymbol{\phi})|n+1\rangle =$$

$$\sum_{j=1}^{n+1} \exp(i\phi_j)\sin\theta_j \prod_{\gamma<j} \cos\theta_\gamma|j\rangle \quad (11)$$

where we have defined  $\theta_{n+1} := \pi/2$  and  $\phi_{n+1} := 0$ . The first  $n$  ones (10) have zero eigenvalue while the last one (11) has eigenvalue  $\varepsilon$ . Notice that for  $n = 1$  the standard 2-level model with the Abelian Berry phase is recovered.

### B. The Connection $A$ and the Field Strength $F$

By using definition (5) the components of the connection  $A$  can be explicitly computed. Their particular form depends on the bundle of the degenerate spaces described in (10) and (11). For  $\mathbf{CP}^n$   $A$  has  $2n$  component as many as the dimensions of the manifold. These  $u(n)$ -valued connection components over  $\mathbf{CP}^n$  are anti-hermitian matrices as dictated by (5). In detail the only non-zero elements of the matrix  $A^{\theta_\beta}$  ( $\beta = 1, \dots, n$ ) are  $A_{\bar{\alpha}\beta}^{\theta_\beta}$  for  $\bar{\alpha} = 1, \dots, \beta-1$ , given by

$$\begin{aligned} A_{\bar{\alpha}\beta}^{\theta_\beta} &= \langle \bar{\alpha} | \mathcal{U}^\dagger \frac{\partial}{\partial \theta_\beta} \mathcal{U} | \beta \rangle \\ &= e^{i(\phi_{\bar{\alpha}} - \phi_\beta)} \sin\theta_{\bar{\alpha}} \prod_{\beta>\gamma>\bar{\alpha}} \cos\theta_\gamma, \end{aligned} \quad (12)$$

as well as  $A_{\bar{\alpha}\beta}^{\theta_\beta} = -A_{\beta\bar{\alpha}}^{\theta_\beta}$  which guarantees the anti-hermiticity. These are  $n$  components corresponding to the  $n$   $\theta$ -coordinates of  $\mathbf{CP}^n$ . The anti-hermitian matrix  $A^{\phi_\beta}$  corresponding to the  $n$   $\phi$ -coordinates has non-zero elements for  $\alpha = \beta$  and  $\alpha \geq \bar{\alpha}$  given by

$$\begin{aligned} A_{\bar{\alpha}\beta}^{\phi_\beta} &= \langle \bar{\alpha} | \mathcal{U}^\dagger \frac{\partial}{\partial \phi_\beta} \mathcal{U} | \beta \rangle \\ &= -ie^{i(\phi_{\bar{\alpha}} - \phi_\beta)} \sin\theta_\beta \sin\theta_{\bar{\alpha}} \prod_{\beta\geq\gamma>\bar{\alpha}} \cos\theta_\gamma, \end{aligned}$$

where we assumed  $\prod_{\beta\geq\gamma>\beta} \cos\theta_\gamma = 1$ , and for  $\beta > \alpha$  and  $\alpha \geq \bar{\alpha}$  by

$$\begin{aligned} A_{\bar{\alpha}\alpha}^{\phi_\beta} &= \langle \bar{\alpha} | \mathcal{U}^\dagger \frac{\partial}{\partial \phi_\beta} \mathcal{U} | \alpha \rangle \\ &= ie^{i(\phi_{\bar{\alpha}} - \phi_\alpha)} \sin\theta_\alpha \sin\theta_{\bar{\alpha}} \sin^2\theta_\beta \prod_{\beta>\gamma>\alpha} \cos\theta_\gamma \prod_{\beta>\bar{\gamma}>\bar{\alpha}} \cos\theta_{\bar{\gamma}}. \end{aligned}$$

Having the transformations  $\mathcal{U}$ , which allow us to fix  $A$ , we are able to determine also the curvature and check the irreducibility properties of the connection for the  $\mathbf{CP}^n$  model. By using the definition (9) and setting  $z_\alpha = z_\alpha^0 + iz_\alpha^1$ , one finds that at  $\mathbf{z} = 0$  the components of the curvature are given by

$$F_{z_\alpha^i z_\beta^j}(0) = \Pi_{\mathbf{z}} \left[ \frac{\partial U_\alpha}{\partial z_\alpha^i}, \frac{\partial U_\beta}{\partial z_\beta^j} \right] \Pi_{\mathbf{z}} \Big|_{\mathbf{z}=0},$$

with  $\alpha, \beta = 1, \dots, n$  and  $i, j = 0, 1$ . The relevant projectors are given by  $\Pi_{\mathbf{z}} = \mathcal{U}(\mathbf{z})\Pi\mathcal{U}(\mathbf{z})^\dagger$ , where  $\Pi$  denotes the projectors over the first  $n$  degenerate eigenstates. Since  $\partial U_\alpha / \partial z_\alpha^i \big|_{\mathbf{z}=0} = i^i (|\alpha\rangle\langle n+1| + (-1)^i |n+1\rangle\langle\alpha|)$ , one finds

$$F_{z_\alpha^i z_\beta^j}(0) = i^{i+j} [(-1)^j |\alpha\rangle\langle\beta| - (-1)^i |\beta\rangle\langle\alpha|].$$

From this expression it follows that the components of  $F$  span the whole  $u(n)$  algebra. As remarked earlier this result does not depend on the specific point chosen, therefore the case considered is irreducible i.e.,  $\Gamma_A(L_{\lambda_0}) \cong U(n)$ . Notice that in order to generate the loops in  $\mathbf{CP}^n$  one needs to control  $2n$  real parameters instead of the  $n^2$  ones necessary for labeling a generic Hamiltonian.

We can now open the way for applying these abstract constructions to quantum computing. In order to generate a given quantum gate  $g \in U(n)$  one has to determine a loop  $C_g$  in  $\mathcal{M} \equiv \mathbf{CP}^n$  such that  $\Gamma_A(C_g) = g$ . As the connection of the  $\mathbf{CP}^n$  model is irreducible, this is possible for any group element  $g$ . Due to the non-Abelian character of the connection such an inverse problem is in general hard to solve. To tackle it we shall take advantage of the composition properties described in the previous section. In particular one chooses specific families of loops  $\{C_i\}$ , that generate a complete set of easily

calculated holonomies, one can eventually construct any  $U(n)$  transformation. To this end first we consider the 2-dimensional sub-manifolds in the  $2n$ -dimensional space  $(\theta, \phi)$ , spanned by two variables,  $(\theta_\beta, \phi_{\bar{\beta}})$  or  $(\theta_\beta, \theta_{\bar{\beta}})$ , for specific values of  $\beta$  and  $\bar{\beta}$ . For these loops the line integral, given by

$$\oint_C A = \oint_C (A^{\theta_\beta} d\theta_\beta + A^{\lambda_{\bar{\beta}}} d\lambda_{\bar{\beta}}), \quad (13)$$

where  $\lambda_{\bar{\beta}} = \theta_{\bar{\beta}}$  or  $\phi_{\bar{\beta}}$ , includes only two of the  $2n$  components of the connection. Of course one cannot just simply calculate this line integral and then exponentiate it because of the path ordering procedure, which is necessary as the matrices  $A^{\theta_\beta}$  and  $A^{\lambda_{\bar{\beta}}}$  in general do not commute with each other. To overcome this difficulty we perform a second step for a further restriction of the loops  $\{C_i\}$ . From (12) one checks that the parameters which define the position of the plane  $(\theta_\beta, \lambda_{\bar{\beta}})$ , where the loop  $C$  lies, can be always chosen in such a way that the matrix  $A^{\theta_\beta}$  is identically zero. In particular, if one takes  $\theta_i = 0, \forall i \neq \beta, \bar{\beta}$ , the matrices  $A^{\theta_\beta}$  and  $A^{\lambda_{\bar{\beta}}}$  commute, so that one can calculate the integral and exponentiate it avoiding the path ordering problem. Of course the

choice of the planes has to be such that the connection components lying on it do not give rise to a trivial holonomy even if they commute with each other. This is indicated by the non-vanishing of the related field strength component  $F_{\theta_\beta \lambda_{\bar{\beta}}}$ . Another interpretation of the holonomy, within this approach, is as the exponential of the flux of  $F$ , through the loop  $C$ . This definition is possible once the problem has been ‘‘Abelianized’’ by having one of the two relevant components of the connection equal to zero. Application of the Stokes theorem then provides a natural way to evaluate the path integral, as  $\oint_C A^{\lambda_{\bar{\beta}}} d\lambda_{\bar{\beta}} = \int_{\mathcal{D}(C)} F_{\theta_\beta \lambda_{\bar{\beta}}} d\theta_\beta d\lambda_{\bar{\beta}}$ , where  $\mathcal{D}(C)$  is the surface the loop  $C$  encloses on the  $(\theta_\beta, \lambda_{\bar{\beta}})$ -plane.

In this framework, it is possible to identify four families of loops in such a way as to produce the basis of four matrices (the Pauli matrices and the identity) of all possible two-by-two sub-matrices belonging to the algebra of  $U(2)$ . With this approach one may restrict to a subspace of the degenerate space spanned by the states  $|\beta\rangle$  and  $|\bar{\beta}\rangle$ , ordered in such a way that  $\beta < \bar{\beta}$ . The relevant sets of coordinates are  $(\theta_\beta, \phi_\beta)$  and  $(\theta_{\bar{\beta}}, \phi_{\bar{\beta}})$ . Taking  $\theta_i = 0$  for all  $i \neq \beta, \bar{\beta}$  one obtains the  $(\theta_\beta, \phi_\beta)$  connection components

$$A^{\theta_\beta} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A^{\phi_\beta} = \begin{bmatrix} -i \sin^2 \theta_\beta & 0 \\ 0 & 0 \end{bmatrix}, \quad (14)$$

while for the  $(\theta_{\bar{\beta}}, \phi_{\bar{\beta}})$  components we have

$$A^{\theta_{\bar{\beta}}} = \begin{bmatrix} 0 & \sin \theta_\beta e^{i(\phi_\beta - \phi_{\bar{\beta}})} \\ -\sin \theta_\beta e^{-i(\phi_\beta - \phi_{\bar{\beta}})} & 0 \end{bmatrix} \quad (15)$$

and

$$A^{\phi_{\bar{\beta}}} = \begin{bmatrix} i \sin^2 \theta_\beta \sin^2 \theta_{\bar{\beta}} & -i \sin \theta_\beta \sin \theta_{\bar{\beta}} \cos \theta_{\bar{\beta}} e^{i(\phi_\beta - \phi_{\bar{\beta}})} \\ -i \sin \theta_\beta \sin \theta_{\bar{\beta}} \cos \theta_{\bar{\beta}} e^{-i(\phi_\beta - \phi_{\bar{\beta}})} & -i \sin^2 \theta_{\bar{\beta}} \end{bmatrix}. \quad (16)$$

With these four matrices we want to build a complete set of generators for the  $U(2)$  group. Specific choices of coordinate planes inside the four dimensional sub-manifold with coordinates  $\{\theta_\beta, \phi_\beta, \theta_{\bar{\beta}}, \phi_{\bar{\beta}}\}$  shall provide those matrices. Note that this sub-manifold is locally isomorphic to  $\mathbf{CP}^2$ , and due to the irreducibility of its

relevant connection it is possible to produce the whole  $U(2)$  group.

We can calculate the components of the field strength associated with the connection components (14),(15), (16). With a straightforward application of (9) we obtain the non-zero field strength components to be

$$F_{\theta_\beta \phi_\beta} = -i \begin{bmatrix} \sin 2\theta_\beta & 0 \\ 0 & 0 \end{bmatrix}, \quad F_{\theta_\beta \phi_{\bar{\beta}}} = i \begin{bmatrix} \sin 2\theta_\beta \sin^2 \theta_{\bar{\beta}} & -\frac{1}{2} \cos \theta_\beta \sin 2\theta_{\bar{\beta}} e^{i(\phi_\beta - \phi_{\bar{\beta}})} \\ -\frac{1}{2} \cos \theta_\beta \sin 2\theta_{\bar{\beta}} e^{-i(\phi_\beta - \phi_{\bar{\beta}})} & 0 \end{bmatrix},$$

$$F_{\theta_\beta \theta_{\bar{\beta}}} = \cos \theta_\beta \begin{bmatrix} 0 & e^{i(\phi_\beta - \phi_{\bar{\beta}})} \\ -e^{-i(\phi_\beta - \phi_{\bar{\beta}})} & 0 \end{bmatrix}, \quad F_{\theta_{\bar{\beta}} \phi_\beta} = -i \sin \theta_\beta \cos^2 \theta_{\bar{\beta}} \begin{bmatrix} 0 & e^{i(\phi_\beta - \phi_{\bar{\beta}})} \\ e^{-i(\phi_\beta - \phi_{\bar{\beta}})} & 0 \end{bmatrix},$$

$$F_{\theta_{\bar{\beta}} \phi_{\bar{\beta}}} = i \begin{bmatrix} 0 & \sin \theta_\beta \cos^2 \theta_{\bar{\beta}} \sin^2 \theta_{\bar{\beta}} e^{i(\phi_\beta - \phi_{\bar{\beta}})} \\ \sin \theta_\beta \cos^2 \theta_{\bar{\beta}} \sin^2 \theta_{\bar{\beta}} e^{-i(\phi_\beta - \phi_{\bar{\beta}})} & -\cos^2 \theta_{\bar{\beta}} \sin 2\theta_{\bar{\beta}} \end{bmatrix}$$

and

$$F_{\phi_\beta \phi_{\bar{\beta}}} = \frac{1}{2} \sin \theta_\beta \cos^2 \theta_{\bar{\beta}} \sin 2\theta_{\bar{\beta}} \begin{bmatrix} 0 & e^{i(\phi_\beta - \phi_{\bar{\beta}})} \\ -e^{-i(\phi_\beta - \phi_{\bar{\beta}})} & 0 \end{bmatrix}.$$

The field strength will be used in the following section to calculate the holonomies.

### C. The Holonomies $\Gamma_A(\gamma)$

Let us see how the restriction on various planes affects the relevant connection components (14),(15),(16). It is logical to choose one of the plane coordinates to be the  $\theta_\beta$  one as  $A_{\theta_\beta} \equiv \mathbf{0}$  and hence it commutes with all others components, as required for implementing our strategy. The first choice is the plane  $(\theta_\beta, \phi_\beta)$ , where the non-zero component of the connection is  $A_{\beta\beta}^{\phi_\beta} = -i \sin^2 \theta_\beta$ . The second choice is the plane  $(\theta_\beta, \phi_{\bar{\beta}})$  for  $\bar{\beta} > \beta$ , with  $\theta_{\bar{\beta}} = \pi/2$ , giving a different connection with two non-zero elements,  $A_{\beta\bar{\beta}}^{\phi_{\bar{\beta}}} = i \sin^2 \theta_\beta$  and  $A_{\bar{\beta}\bar{\beta}}^{\phi_{\bar{\beta}}} = -i$ . Of course the latter element will give zero when integrated along a loop. For  $\beta > \bar{\beta}$  both matrices are identically zero, and give rise to a trivial holonomy. With these two components and for appropriate loops one can obtain all possible  $U(n)$  diagonal transformations. Indeed, for the loop  $C_1 \in (\theta_\beta, \phi_\beta)$  we obtain

$$\Gamma_A(C_1) = \exp[-i|\beta\rangle\langle\beta|\Sigma_1]$$

$\Sigma_1$  denoting the area enclosed by  $C_1$  on a  $S^2$  sphere with coordinates  $(2\theta_\beta, \phi_\beta)$ . This is exactly the Abelian Berry phase which could be produced if the state  $|\beta\rangle$  were non-degenerate as it does not get mixed with the rest of the states. For  $C_2 \in (\theta_\beta, \phi_{\bar{\beta}})$  we obtain analogously the holonomy

$$\Gamma_A(C_2) = \exp[i|\beta\rangle\langle\beta|\Sigma_2] .$$

Recalling the constraint  $\beta < \bar{\beta}$ , we see that one can produce  $n-1$  distinct holonomies from  $C_2$  type loops.

To obtain the non-diagonal transformations one has to consider a loop on the  $(\theta_\beta, \theta_{\bar{\beta}})$  plane, with  $\theta_i = 0$  for all  $i \neq \beta, \bar{\beta}$ . Then, the only non-vanishing elements of the connection are  $A_{\beta\bar{\beta}}^{\theta_{\bar{\beta}}} = e^{i(\phi_\beta - \phi_{\bar{\beta}})} \sin \theta_\beta = -A_{\bar{\beta}\beta}^{\theta_{\bar{\beta}}}$ . By choosing further the  $(\theta_\beta, \theta_{\bar{\beta}})$  plane at the position  $\phi_\beta = \phi_{\bar{\beta}} = 0$  the holonomy becomes, for the loop  $C_3 \in (\theta_\beta, \theta_{\bar{\beta}})_{\phi_\beta=\phi_{\bar{\beta}}=0}$

$$\Gamma_A(C_3) = \exp[-i(-i|\beta\rangle\langle\bar{\beta}| + i|\bar{\beta}\rangle\langle\beta|)\tilde{\Sigma}_3] ,$$

while at  $\phi_\beta = \pi/2$  and  $\phi_{\bar{\beta}} = 0$ , i.e.  $C_4 \in (\theta_\beta, \theta_{\bar{\beta}})_{\phi_\beta=\pi/2, \phi_{\bar{\beta}}=0}$  we have

$$\Gamma_A(C_4) = \exp[-i(|\beta\rangle\langle\bar{\beta}| + |\bar{\beta}\rangle\langle\beta|)\tilde{\Sigma}_4] ,$$

where  $\tilde{\Sigma}$  is the area on the sphere with coordinates  $(\pi/2 - \theta_\beta, \theta_{\bar{\beta}})$ . The positive or negative sign in front of the area depends on the orientation of the surface enclosed by the loop  $C$  with respect to the orientation of the field strength  $F$ . Note that any loop  $C$  on the  $(\theta_\beta, \lambda_{\bar{\beta}})$  plane

with the same enclosed area  $\Sigma_C$  (when mapped on the appropriate sphere) will give the same holonomy independent of its position and shape. These four holonomies restricted each time to a specific  $2 \times 2$  sub-matrix generate all  $U(2)$  transformations. Hence, considering the inverse problem of obtaining a desired unitary from a holonomy we are able to choose from a whole family of loops of arbitrary shape and position.

Finally, it is easy to check that in this way one can indeed obtain  $U = \exp[\mu_a T_a]$ , where  $T_a$  ( $a = 1, \dots, n^2$ ) is a  $u(n)$  anti-hermitian generator and  $\mu_a$  an arbitrary real number. Therefore any element of  $U(n)$  can be obtained by controlling the  $2n$  parameters labeling the points of  $\mathbf{CP}^n$ .

It is instructive to consider the form that the Hamiltonian family  $\mathcal{F}$  takes when restricted to the particular 2-sub-manifolds. For the loop  $C_1$  (and similarly for  $C_2$ ) one finds

$$H_1 = -\varepsilon/2 \vec{B}(2\theta_\beta, \phi_\beta) \cdot \vec{\sigma}$$

for  $\vec{B}(\theta_i, \phi_j) = (\sin \theta_i \cos \phi_j, \sin \theta_i \sin \phi_j, \cos \theta_i)^T$ , where the only non-zero elements are on the  $\beta$ -th and  $(n+1)$ -th row and column.  $H_1$  generates an Abelian  $\mathbf{CP}^1$  phase in front of the state  $|\beta\rangle$  and the conjugate one in front of  $|n+1\rangle$ . On the other hand for the path  $C_3$  (and similarly for  $C_4$ ) we have

$$H_3 = \varepsilon \vec{B}(\theta_\beta, \theta_{\bar{\beta}}) \vec{B}(\theta_\beta, \theta_{\bar{\beta}})^T ,$$

where the non-zero elements connect the states  $|\beta\rangle$ ,  $|\bar{\beta}\rangle$  and  $|n+1\rangle$ . In this Hamiltonian there is direct coupling between three states, giving rise to a non-Abelian interaction. While  $H_1$  is easy to simulate in the laboratory with various experimental setups (spin 1/2 particle in a magnetic field, NMR, optical polarization, etc) it is yet an open challenge to construct the interaction dictated by the Hamiltonian  $H_3$ .

## IV. APPLICATION TO THE OPTICAL HOLONOMIC SETUP

In the previous sections we have developed the theoretical background in order to make the non-Abelian geometrical phases useful into the quantum computing arena. Complex as it may be, such a construction offers various possibilities and advantages when applied to physical systems. In particular we shall resort to quantum optics in order to make a physical application of HQC [9], but also to clarify and resolve some theoretical issues discussed in the previous sections. To this aim we employ existing devices of quantum optics, such as displacing and squeezing devices and interferometers, acting on laser beams in a non-linear medium.



### A. Displacers, Squeezers and Interferometers as holonomic devices

In this subsection we shall consider the realistic implementation of the holonomic computation in the frame of quantum optics. Even though the complete implementation of the model presented here is likely to be experimentally a very challenging task, it is still remarkable that the necessary employed devices are realizable in the laboratory. Let us briefly present the setup.

In order to perform holonomic computation with laser beams we shall consider the non-linear interaction Hamiltonian produced by a Kerr medium

$$H_I = \hbar X n(n-1) ,$$

with  $n = a^\dagger a$  the number operator,  $a$  and  $a^\dagger$  being the usual bosonic annihilation and creation operators respectively, and  $X$  a constant proportional to the third order nonlinear susceptibility,  $\chi^{(3)}$ , of the medium. The degenerate eigenstates of  $H_I$  are the  $|0\rangle$  and  $|1\rangle$ , where  $\{|\nu\rangle; \nu = 0, 1, \dots\}$  denote the Fock basis of number eigenstates,  $n|\nu\rangle = \nu|\nu\rangle$ . The degenerate space they span will

$$A_x = \begin{bmatrix} -iy & -(\cosh 2r_1 - e^{i\theta_1} \sinh 2r_1) \\ \cosh 2r_1 - e^{-i\theta_1} \sinh 2r_1 & -iy \end{bmatrix} ,$$

$$A_y = \begin{bmatrix} ix & i(\cosh 2r_1 + e^{i\theta_1} \sinh 2r_1) \\ i(\cosh 2r_1 + e^{-i\theta_1} \sinh 2r_1) & ix \end{bmatrix} ,$$

$$A_{r_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \quad A_{\theta_1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{i}{4} (\cosh 4r_1 - 1) .$$

The diagonal elements of the above matrices are also given easily by the equivalent Berry phases acquired by non-degenerate Fock states  $|\nu\rangle$ . For example the Abelian phases produced by a displacer are equal for all Fock states and given by [19]

$$\phi_{Berry}^\nu = \oint (ydx - xdy) , \quad (17)$$

while the phases produced by a squeezer are given by

$$\phi_{Berry}^\nu = \frac{2\nu+1}{4} \oint (\cosh 4r_1 - 1) d\theta_1 .$$

Notice the complete similarity of the generators of the squeezing phases with  $A_{\theta_1}$  as the non-Abelian matrix is diagonal, while in the displacing case only the diagonal elements are reproduced by (17).

On the other hand the parametric space of the two mode interferometers gives rise to the following connection components

be the one qubit coding space. For the tensor product structure of  $l$  qubits we have to employ a set of  $l$  beams, providing the basis states  $|\nu_1 \dots \nu_m\rangle = |\nu_1\rangle \otimes \dots \otimes |\nu_m\rangle$  where  $\nu_l$  could be zero or one, for  $l = 1, \dots, m$ . The code can be written in this space of states.

The iso-spectral transformations of the Hamiltonian,  $H(\sigma) = \mathcal{U}(\sigma) H_I \mathcal{U}^\dagger(\sigma)$ , can be constructed by resorting to displacing and squeezing devices with unitaries  $D(\lambda) = \exp(\lambda a^\dagger - \bar{\lambda} a)$  and  $S(\mu) = \exp(\mu a^{\dagger 2} - \bar{\mu} a^2)$  respectively, as well as two mode displacing and squeezing devices:  $N(\xi) = \exp(\xi a_1^\dagger a_2 - \bar{\xi} a_1 a_2^\dagger)$  and  $M(\zeta) = \exp(\zeta a_1^\dagger a_2^\dagger - \bar{\zeta} a_1 a_2)$  respectively. In fact to obtain holonomies for one qubit gates we employ  $\mathcal{U}(\sigma) = D(\lambda) S(\mu)$  while for two qubits we take  $\mathcal{U}(\sigma) = N(\xi) M(\zeta)$ . In the first case the connection components are two-by-two matrices, where the null action is assumed on the rest of the tensor product sub-systems. For the coordinate decomposition  $\lambda = x + iy$  and  $\mu = r_1 \exp i\theta_1$  we have after some simple algebra, similar to the one for  $\mathbf{CP}^2$ , the connection components

$$A_{r_2} = \begin{bmatrix} 0 & 0 & 0 & -e^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{bmatrix} ,$$

$$A_{r_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-i\theta_3} & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (2 \cosh^2 r_2 - 1) .$$

as well as

$$A_{\theta_2} = \begin{bmatrix} 0 & 0 & 0 & e^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{bmatrix} \frac{i}{2} \sinh 2r_2 +$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \frac{i}{2} (\cosh 2r_2 - 1) ,$$

and

$$A_{\theta_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\theta_3} & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{i}{2} \cosh 2r_2 \sin 2r_3 +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} i \sin^2 r_3 ,$$

where  $\zeta = r_2 \exp i\theta_2$  and  $\xi = r_3 \exp i\theta_3$ . Here we define  $\mathcal{M} := \{\sigma_i\}$  th  $\sigma_i$ 's being the real coordinates  $\{x, y, r_i, \theta_i\}$  parameterizing all possible configurations with devices acting on the various laser beams. The first two components are enough for constructing holonomies representing two qubit gates between any two qubits, which together with the one qubit rotations result into a universal set of gates. It is straightforward to apply the conditions posed in the previous section for finding coordinate planes, with the relevant connection components commuting. As  $A_{r_1}$  is identically zero it can be combined with  $A_x$ ,  $A_y$  and  $A_{\theta_1}$  to produce  $U(2)$  holonomies, while  $A_{r_2}$  and  $A_{r_3}$  commute with each other leading to generate  $U(4)$  matrices.

The holonomies obtained from the various loops are given in the following. The loop  $C_I \in (x, r_1)_{\theta_1=0}$  gives

$$\Gamma_A(C_I) = \exp -i\hat{\sigma}_1 \Sigma_I$$

with  $\Sigma_I := \int_{D(C_I)} dx dr_1 2e^{-2r_1}$ . The loop  $C_{II} \in (y, r_1)_{\theta_1=0}$  gives

$$\Gamma_A(C_{II}) = \exp -i\hat{\sigma}_2 \Sigma_{II} ,$$

with area  $\Sigma_{II} := \int_{D(C_{II})} dy dr_1 2e^{2r_1}$ . The loop  $C_{III} \in (r_1, \theta_1)$  gives

$$\Gamma_A(C_{III}) = \exp -i\hat{s}_3 \Sigma_{III} ,$$

with  $\Sigma_{III} := \int_{D(C_{III})} dr_1 d\theta_1 \sinh 4r_1$ .  $C_I$ ,  $C_{II}$  and  $C_{III}$  produce  $U(2)$  rotations. In order to generate  $U(4)$  group elements we span the following configurations. The loop  $C_{IV} \in (r_2, r_3)_{\theta_2=\theta_3=0}$  gives

$$\Gamma_A(C_{IV}) = \exp -i\hat{\sigma}_2^{12} \Sigma_{IV}$$

with  $\Sigma_{IV} := \int_{D(C_{IV})} dr_2 dr_3 2 \sinh 2r_2$ . The loop  $C_V \in (r_2, r_3)_{\theta_2=0, \theta_3=3\pi/2}$  gives

$$\Gamma_A(C_V) = \exp -i\hat{\sigma}_1^{12} \Sigma_V$$

with the area given by  $\Sigma_V := \int_{D(C_V)} dr_2 dr_3 2 \sinh 2r_2$ . Above we have used

$$\hat{s}_3 := - \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} , \quad \hat{\sigma}_2^{12} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\hat{\sigma}_1^{12} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ,$$

while  $D(C_\rho)$  with  $\rho = I, \dots, V$  is the surface on the relevant sub-manifold  $(\sigma_i, \sigma_j)$  of  $\mathcal{M}$  whose boundary is the path  $C_\rho$ . The hyperbolic functions in these integrals stem out of the geometry of the  $su(1,1)$  manifold associated with the relevant control sub-manifold. The  $\Gamma_A(C)$ 's thus generated belong either to the  $U(2)$  or  $U(4)$  group and act on the one qubit space or on the space of the tensor product of two qubits, respectively. Considering the tensor product structure of our system these rotations represent in the  $2^m$  space of  $m$  qubits respectively single qubit rotations and two qubit interactions, thus resulting into a universal set of logical gates. For example,  $\Gamma_A(C_V)$  with  $\Sigma_V = \pi/4$  gives the following non-trivial two qubits gate

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

while the holonomies produced by the loops  $C_I$ ,  $C_{II}$  and  $C_{III}$  give a general one qubit rotation. Together they compose a universal set of transformations [20].

The above description can be applied also to quantum systems with harmonic oscillator structure. For example, ion traps or atoms in a cavity are described by oscillating modes (the ionic vibration or the cavity mode), which can be also displaced, squeezed and interfered with standard experimental techniques [21–24]. The degeneracy can be provided by the coupling of the oscillating modes with the internal energy levels of the atoms or by the structure of the product space of states of two modes [25].

## B. $SU(2)$ Interferometers and non-Abelian Stokes Theorem

In what follows we discuss the possible use of  $SU(2)$  interferometer as a control devices [26] for producing holonomies. While relevant to quantum optical devices, such a scheme has an additional theoretical interest. Rather than distilling the Abelian sub-structure to avoid the path ordering problem we shall employ the non-Abelian Stokes theorem in order to rewrite the loop integral of the connection as a surface integral of its field strength. The main advantage of this approach is the partial relaxation of the path ordering conditions, enabling

the holonomic calculation of non-commuting connection components.

For  $a_1$  and  $a_2$  the annihilation operator of two different laser beams, consider the Hermitian operators

$$\begin{aligned} J_x &= \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), \quad J_y = -\frac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1), \\ J_z &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \end{aligned} \quad (18)$$

and

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 = n_1 + n_2.$$

The operators (18) satisfy the commutation relations for the Lie algebra of  $SU(2)$ ;  $[J_x, J_y] = iJ_z$ ,  $[J_y, J_z] = iJ_x$ ,  $[J_z, J_x] = iJ_y$ . The operator  $N$ , which is proportional to the free Hamiltonian of two laser beams, commutes with all of the  $J$ 's. On the other hand, however, the Kerr Hamiltonian does not commute with the  $J$ 's, allowing for the possibility that  $SU(2)$  interferometers be used as transformation controllers in view of the holonomic computation.

From the above operators we obtain the unitaries,  $U_x(\alpha) = \exp(i\alpha J_x)$ ,  $U_y(\beta) = \exp(i\beta J_y)$  and  $U_z(\gamma) = \exp(i\gamma J_z)$ . For the degenerate state space of two laser beams spanned by  $|\nu_1 \nu_2\rangle$ , we have from (5) and for  $\mathcal{U} = U_x(\alpha)U_y(\beta)U_z(\gamma)$  the following connection components

$$A_\alpha = \frac{i}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin \beta & \cos \beta e^{i\gamma} & 0 \\ 0 & \cos \beta e^{-i\gamma} & -\sin \beta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_\beta = -\frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\gamma} & 0 \\ 0 & -e^{-i\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A_\gamma = -\frac{i}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

These components do not commute with each other, when projected on planes with non-trivial field strength. Hence, it is not possible to adopt again the method used previously to calculate the holonomies of paths in the three dimensional parameter space,  $(\alpha, \beta, \gamma)$ . Instead, for this purpose we can employ the non-Abelian Stokes theorem [27]. The extra limitation, now, for the choice of the path comes from the constraint that, apart from being confined on a special two dimensional subspace, it has to have the shape of an orthogonal parallelogram with two sides lying along the coordinate axis. This will

facilitate the extraction of an analytic result from the Stokes theorem. With the loop composition properties described earlier, it is possible to generalize these simple structures to any desirable loop.

In order to state the non-Abelian Stokes theorem let us first present some preliminaries, where a few simplifications are introduced, as its general form will not be necessary in the present work. Consider the Wilson loop (holonomy),  $W = \mathbf{P} \exp \oint_C A$ , of a rectangular loop  $C$  with the sides parallel to the coordinates  $(\sigma, \tau)$  where  $\sigma$  and  $\tau$  are a parameterization of the plane where the loop  $C$  lies.  $W$  is made out of the Wilson lines  $W_i$  for  $i = 1, \dots, 4$ , as  $W = W_4 W_3 W_2 W_1$ , where  $W_i$  corresponds to the  $i$ 'th side of the rectangular ordered in anti-clockwise way. Define  $T^{-1}(\sigma, \tau) \equiv W_4 W_3$ . Then, for  $F_{\sigma\tau}$  the field strength of the connection  $A$  on the plane  $(\sigma, \tau)$ ,  $W$  is given in terms of a surface integral

$$W = \mathbf{P}_\tau e^{\int_\Sigma T^{-1}(\sigma, \tau) F_{\sigma\tau}(\sigma, \tau) T(\sigma, \tau) d\sigma d\tau}, \quad (19)$$

where  $\mathbf{P}_\tau$  is the path ordering symbol with respect only to the  $\tau$  variable contrary to the usual path ordering symbol  $\mathbf{P}$  which is defined with respect to both variables,  $\sigma$  and  $\tau$ . Here  $F_{\sigma\tau}(\sigma, \tau) = -\partial_\sigma A_\tau + \partial_\tau A_\sigma + [A_\sigma, A_\tau]$ .

The key point of this approach is that in (19) the path ordering is only with respect to  $\tau$  so there is the possibility to arrange the exponent in such a way that its integration with respect to  $\tau$  is performed with a trivial path ordering procedure. Such possibility can be achieved in the present case by choosing the integration limits with respect to  $\sigma$  so that they give commuting matrices as functions of  $\tau$ . In detail, we first choose a loop  $C_1$  on the plane  $(\alpha, \beta)$  positioned at  $\gamma = 0$ . The relevant field strength component and the matrix  $T^{-1}(\alpha, \beta)$  are given by

$$F_{\alpha\beta} = i(\cos \beta \hat{\sigma}_3 - \sin \beta \hat{\sigma}_1)$$

and

$$T^{-1} = \exp(i\frac{\beta}{2}\hat{\sigma}_2) \exp\left[-i\frac{\alpha}{2}(\sin \beta \hat{\sigma}_3 + \cos \beta \hat{\sigma}_1)\right].$$

Hence we obtain

$$\begin{aligned} T^{-1} F_{\alpha\beta} T &= \\ -i(\cos \alpha \sin 2\beta \hat{\sigma}_1 - \cos \alpha \cos 2\beta \hat{\sigma}_3 + \sin \alpha \hat{\sigma}_2) \end{aligned} \quad (20)$$

By taking a rectangle with the  $\alpha$ -side equal to  $\pi$  the only remaining term after the  $\alpha$  integration is the one with  $\hat{\sigma}_2$ , and hence the path ordering with respect to  $\beta$  does not pose any computational problem. As a result we have that for the closed rectangular loop  $C_1 \in (\alpha, \beta)$ -plane with coordinates  $\{(0, 0), (\alpha = \pi, 0), (\alpha = \pi, \beta), (0, \beta)\}$  we obtain the following unitary transformation

$$\Gamma_A(C_1) = \exp(-i2\beta \hat{\sigma}_2^{12}). \quad (21)$$

With similar reasoning a rectangular loop  $C_2 \in (\alpha, \gamma)$ -plane with coordinates  $\{(0, 0), (\alpha = \pi, 0), (\alpha = \pi, \gamma), (0, \gamma)\}$  has the field strength component and the matrix  $T^{-1}(\alpha, \gamma)$  given by

$$F_{\alpha\gamma} = -i(\cos \gamma \hat{\sigma}_2 + \sin \gamma \hat{\sigma}_1)$$

and

$$T^{-1} = \exp(i\frac{\gamma}{2}\hat{\sigma}_3) \exp\left[-i\frac{\alpha}{2}(\cos \gamma \hat{\sigma}_1 - \sin \gamma \hat{\sigma}_2)\right],$$

which give finally the holonomy

$$\Gamma_A(C_2) = \exp(-i2\gamma\hat{\sigma}_3^{12}).$$

The matrix  $\hat{\sigma}_3^{12}$  is defined similarly to  $\hat{\sigma}_1^{12}$  and  $\hat{\sigma}_2^{12}$  in Subsection IV A. Note that the coefficients in front of the matrices in the unitaries are areas on spheres spanned by the angles  $\alpha$  and  $\beta$  or  $\alpha$  and  $\gamma$ . This is consistent with the geometry of  $SU(2)$ .

Due to their structure the two matrices  $\Gamma_A(C_1)$  and  $\Gamma_A(C_2)$  can produce any unitary transformation of *one* qubit encoded in the sub-space of states of two laser beams spanned by  $\{|01\rangle, |10\rangle\}$ . In other words, we need two laser beams to encode one qubit, contrary to previous construction. The operation of two qubit interactions demands sophisticated optical devices as four mode interferometers, and we shall not discuss them here.

### C. Holonomies and Devices

In the optical example studied above, a general state  $|\psi\rangle$  in the degenerate eigen-space of  $H_0 = H_{Kerr}$  is given as a linear combination of  $|0\rangle$  and  $|1\rangle$ . The interaction Hamiltonian  $H_{Kerr}$  describes fully our system when we move to the interaction picture. This transition affects only the parameters of the control devices which are re-defined. Under an iso-spectral, cyclic and adiabatic evolution of the Hamiltonian in the family  $\mathcal{F}$ , the evolution operator acting on  $|\psi\rangle$  is given by the  $2 \times 2$  sub-matrix in the upper left corner of the matrix

$$U(0, T) = \mathbf{T} \exp -i \int_0^T \mathcal{U}(\sigma(t)) H_0 \mathcal{U}^\dagger(\sigma(t)) dt. \quad (22)$$

This evolution takes place from time 0 to time  $T$  and, to perform a closed path  $\gamma$ , we demand  $\sigma(0) = \sigma(T)$ . The iso-spectral rotations of the Hamiltonian can be achieved with the use of the optical control devices described in the previous subsections. The degeneracy is provided by the propagation of the laser beams through the Kerr medium. Hence, when the laser beams are in the control devices, the degeneracy is not present. To restore degeneracy along the adiabatic evolution one may employ the kicking method [8], [18] now briefly recalled.

Suppose that one is able to turn on and off a set of interactions (the “kicks”)  $K := \{\mathcal{U}(\sigma)\}_{\sigma \in \mathcal{M}}$  on an

ultra-fast time-scale with respect to the unperturbed part of the evolution. Let  $T = N \Delta t$  and  $t_0 = 0, t_{i+1} = t_i + \Delta t (i = 1, \dots, N-1)$  be a partition of the time interval  $[0, T]$ . The system evolution is as follows: at any time  $t_i$  one kicks the system with the pulse  $\mathcal{U}_{i+1}^\dagger \mathcal{U}_i$  where  $\mathcal{U}_i := \mathcal{U}(\sigma_i)$  is a unitary belonging to  $K$  ( $\mathcal{U}_0 = \mathcal{U}_N = \mathbb{1}$ ). If between the kicks the system evolution is unperturbed, one obtains the evolution along the loop  $\gamma$  by

$$U_\gamma = \mathbf{T} \prod_{i=1}^{N-1} \mathcal{U}_i e^{-iH_0 \Delta t} \mathcal{U}_i^\dagger,$$

In the limit  $\Delta t \mapsto 0, N \mapsto \infty, [N \Delta t = T]$  one gets  $U_\gamma \rightarrow \mathbf{T} \exp \int_0^T dt H(t)$  where  $H(t) := \mathcal{U}(\sigma(t)) H_0 \mathcal{U}(\sigma(t))^\dagger$ . By making the function  $\sigma(t)$  vary adiabatically, one can approximate the holonomic evolution. The latter simulates the desired evolution (22) by alternating the action of the degenerate Hamiltonian,  $H_0$ , and the control procedure with infinitesimal steps. We shall employ it here to demonstrate in a simple example the fidelity of the evolution operator approximated by the kicking method procedure and the relevant holonomy theoretically predicted.

Let us consider displacing devices  $D(\lambda)$ , performing a closed loop in their control parameters  $\lambda$ .

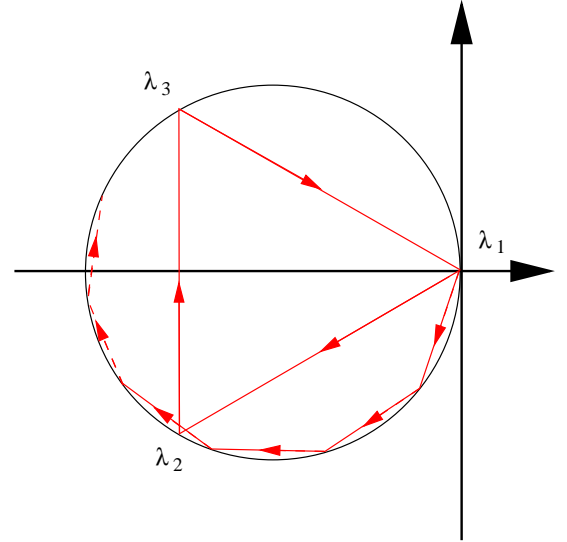


FIG. 1. The triangular (and polygonal) loop  $C$  on the complex plane of the displacing control parameters,  $\lambda$ , approximating the loop (circle).

This is shown in Fig. 1, where for simplicity the least possible number of displacing devices (three) for performing a closed loop has been drawn. When the state is at the edges of the polygon (triangle) no displacement takes place and the action of the Kerr medium is implied. Two displacing unitaries are combined as  $D(\lambda)D(\lambda') = \exp(i\Im(\lambda\bar{\lambda}'))D(\lambda + \lambda')$ . The physical process behind this is as follows. On the state  $|\psi\rangle$  first

acts a displacing unitary  $D^\dagger(\lambda_1)$ , taking it to the point  $\lambda_1$ . Then, the evolution operator of the Kerr Hamiltonian,  $U(\Delta t) = \exp(-iH_0\Delta t)$ , acts for a time interval  $\Delta t = T/3$ . This effect is achieved by propagating the beam inside a Kerr medium. Then, the evolution  $D^\dagger(\lambda_2)D(\lambda_1)$  is performed. This is obtained, with a single displacing device, given (up to an overall phase factor that will cancel at the end) by  $D(\lambda_1 - \lambda_2)$ . After exiting the displacing device (we are at point  $\lambda_2$ ) the beam en-

ters a Kerr medium once more for time  $\Delta t$  and then the procedure is repeated until one comes back to the point  $\lambda_1$  and the beam enters again the Kerr medium. Eventually, the state is thus displaced by  $D(\lambda_1)$ . This loop may be transported to any other place of the control parameter complex plane by acting at the beginning and at the end of this procedure with the appropriate displacing unitary (device).

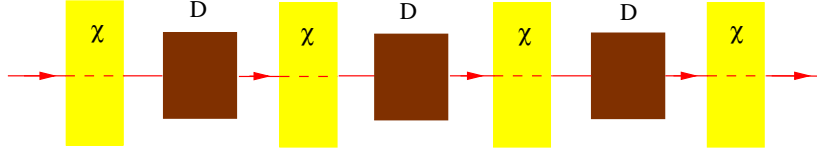


FIG. 2. The setup with the alternating Kerr media and Displacing devices.

The evolution operator is approximated by the following evolution operator

$$U(0, T) \approx D(\lambda_1) \left( U(\Delta t; 0) U(\Delta t; \tilde{\lambda}_1 + \tilde{\lambda}_2) \right. \\ \left. U(\Delta t; \tilde{\lambda}_1) U(\Delta t; 0) \right) D^\dagger(\lambda_1),$$

where  $U(\Delta t; \tilde{\lambda}) = D(\tilde{\lambda})U(\Delta t)D^\dagger(\tilde{\lambda})$ ,  $\tilde{\lambda}_i = \lambda_{i+1} - \lambda_i$  and  $\lambda_4 = \lambda_1$ . For integer  $N$  It is instructive to simulate numerically the above operator studying its dependence on the number of discrete steps used to approximate the continuous procedure. We start with five kicks. The involved parameters are taken to be  $T = 0.1$  and  $\hbar X = 1$ ,

with the radius of the circle equal to 1. The initial point is taken to be the origin of the complex plane rather than  $\lambda_1$ , or in other words we do not perform the initial and final displacements by  $D^\dagger(\lambda_1)$  and  $D(\lambda_1)$ . In the table below are given the percentage deviations of the absolute values of the  $(0,0)$ ,  $(0,1)$ ,  $(1,0)$ ,  $(1,1)$  elements of the evolution operator  $U(0, T)$  as functions of the number of displacers used to approximate the cyclic evolution. These are the relevant elements for the evolution of the states in the degenerate eigen-space describing a qubit. Numerical values were obtained for 5, 10, 20 and 26 displacers and the percentage is with respect to the value obtained with 100 displacers.

	5	10	20	26
00	0.2419 %	0.0595 %	0.0149 %	0.0099 %
01	0.9119 %	0.2260 %	0.0558 %	0.0186 %
10	0.9119 %	0.2260 %	0.0558 %	0.0186 %
11	1.6763 %	0.4061 %	0.0760 %	0.0269 %

We see that with 26 displacers the error is of the order of 1 in  $10^4$  acceptable for quantum computation with error correction. This provides an indication for the necessary number of devices needed in order to reproduce faithfully the holonomic adiabatic loop.

In the polygon of Fig. 1 the displacement from point  $\lambda_i$  to  $\lambda_{i+1}$  is performed by the operating device with unitary action  $D(\lambda_{i+1} - \lambda_i)$ . This device is naturally affected by an error in the position of the point  $\lambda_{i+1}$ . As an overall effect this will introduce an error in the area of the enclosed surface bounded by the loop  $C$ , as well as an error in the “matching” of the initial and final points  $\lambda_1$  and  $\lambda_{N+1}$ . The enclosed area is the parameter of the unitaries interpreted as logical gates, while the matching of the points  $\lambda_1$  and  $\lambda_{N+1}$  is required in order to apply the adiabatic theorem for the holonomic interpretation of the gates. In both cases the error is zero in the first

order (i.e. our model is robust) with respect to the error introduced by spanning the loop  $C$ .

Another appealing characteristic of the holonomic model is the following. Consider  $\tau_k$  to be the (kicking) time during which each displacer acts,  $T$  the overall period of the evolution and  $\omega$  the characteristic frequency of our system with Hamiltonian  $H_0$ . Then, in order to apply the adiabatic theorem and the kick method the following inequality should hold

$$\tau_k \ll \omega^{-1} \ll T$$

This condition is robust against small variations of the parameters  $\tau_k$  and  $T$ .

## V. CONCLUSIONS

In this paper we provided a detailed account of the so-called Holonomic approach to quantum computation (HQC). While in the standard “dynamical” picture the information processing is performed by a sequence (network) of logic gates obtained by turning on and off suitable Hamiltonians in HQC the idea is to exploit the tools of non-abelian gauge theories to manipulate quantum information. The quantum codewords are realized by  $n$ -fold degenerate eigenstates of an iso-degenerate  $d$ -parametric family of Hamiltonians  $\mathcal{F} = \{H(\lambda)\}_\lambda$ . The quantum gates are then given by the holonomies  $U(\gamma) \in U(n)$  produced by moving along loops in the manifold  $\mathcal{M}$  of control parameters  $\lambda$ . The holonomies represent the non-commutative generalization of the well-known Berry geometric phases and their existence is due to the non-trivial geometry of the set of the computational eigenspaces thought of as a complex vector bundle over the control manifold  $\mathcal{M}$ .

We focused on the case in which these loops are traveled in an adiabatic way. In this situation one can find explicit expression for the non-abelian ( $u(n)$ -valued) connection i.e., gauge potential, form  $A$  such that  $U(\gamma) = \mathbf{P} \exp \int_\gamma A$ . The computational power of the given connection  $A$  is described in terms of the associated curvature form  $F$ : when  $d(d-1)/2$  components of  $F$  span the whole  $u(n)$  then one can perform universal QC over the code. We thoroughly discussed several examples showing explicitly how to design loops, in the relevant control manifolds, in such a way as to get universal set of gates. We argued how, by using suitable Holonomic families  $\mathcal{F}$  acting on multi-partite systems, one can achieve efficient computation taking advantage of quantum entanglement. We also discussed a potential experimental demonstration of HQC in a quantum-optical set-up, using artificial Holonomic family  $\mathcal{F}$  generated with an iterated “kicks” method.

The HQC approach shows how many of the notions and techniques developed in (non-abelian) gauge theories can find a natural application and interpretation in the arena of quantum information processing. This rather unexpected connection between the tools used for the description of Nature at its fundamental level and the ideas of Information/Computation theory is conceptually quite intriguing. In a slogan HQC suggest that information, besides being “physical” it can also be “geometrical”. On the more concrete side HQC, similarly to the other geometrical/topological approaches to QC recently emerged, shows inherent fault-tolerant issues. This latter fact could make HQC appealing even from the implementation point of view, in spite of the quite demanding requirements it involves. Individuation of potential candidates for realization of HQC schemes in the lab is a major challenge for future research.

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- [1] For reviews, see D.P. DiVincenzo and C. Bennet *Nature* **404**, 247 (2000); A. Steane, *Rep. Prog. Phys.* **61**, 117 (1998)
  - [2] R. Jozsa, *Entanglement and Quantum Computation*, quant-ph/9707034
  - [3] A. Kitaev, *Fault-Tolerant computations with Anyons*, quant-ph/9707021; *Fermionic quantum computation* quant-ph/0003137 ;
  - [4] M. Freedman et al, *Simulation of topological field theories by quantum computers*, quant-ph/0001071; *A modular functor which is universal for quantum computation*, quant-ph/0001108; *Quantum Computation and the localization of Modular Functors*, quant-ph/0003128.
  - [5] A. Ekert et al, *Geometric Quantum Computation*, quant-ph/0004015; J. A. Jones et al, *Nature* **403**, 869 (2000).
  - [6] S. Lloyd, *Quantum computation with Abelian Anyons*, quant-ph/0004010.
  - [7] P. Zanardi and M. Rasetti, *Phys. Lett. A* **264** (1999) 94, quant-ph/9904011.
  - [8] J. Pachos, P. Zanardi, and M. Rasetti, *Phys. Rev. A* **61**,010305(R) (2000), quant-ph/990710.
  - [9] J. Pachos and S. Chountasis, *Optical Holonomic Quantum Computer*, quant-ph/9912093 .
  - [10] J. Pachos, *Quantum Computation by Geometrical Means*, to be published in the AMS Contemporary Math Series volume entitled *Quantum Computation and Quantum Information Science*, quant-ph/0003150.
  - [11] K. Fujii, *Note on Coherent States and Adiabatic Connections, Curvatures* , quant-ph/9910069; *Mathematical Foundations of Holonomic Quantum Computer*, quant-ph/0004102.
  - [12] T.T. Wu and C. N. Yang, *Phys. Rev. D* **12**, 3845 (1975), R. Jackiw in *Relativity Groups and Topology*, Les Houches 1983, 154pp, B.S. DeWitt and R. Stora Eds. North-Holland, 1984.
  - [13] D. Deutsch, A. Barenco and A. Ekert, *Proc. R. Soc. London A*, **449**, 669 (1995); D.P. Di Vincenzo, *Phys. Rev. A*, **50**, 1015 (1995).
  - [14] F. Wilczek and A. Zee., *Phys. Rev. Lett.* **52**, 2111 (1984).
  - [15] M. Nakahara, *Geometry, Topology and Physics*, IOP Publishing Ltd., 1990.
  - [16] For a review see, *Geometric Phases in Physics*, A. Shapere and F. Wilczek, Eds. World Scientific, 1989.
  - [17] J. Preskill, *Fault-tolerant quantum computation* in *Introduction to quantum computation and information*, Hoi-Kwong Lo, S. Popescu and T. Spiller Eds., World Scientific, Singapore, 1999.
  - [18] P. Zanardi, *Phys. Lett. A* **258**, 77 (1999).

- [19] S. Seshadri, S. Lakshmibala and V. Balakrishnan, quant-ph/9905101.
- [20] D. Loss and D. DiVincenzo, Phys. Rev. A **57**, 120 (1998).
- [21] D. Meekhof, C. Monroe, B. King, W. Itano and D. Wineland, Phys. Rev. Lett. **76**, 1796 (1996).
- [22] J. Cirac, A. Parkins, R. Blatt and P. Zoller, Phys. Rev. Lett. **70**, 556 (1993).
- [23] J. Steinbach, J. Twamley and P. Knight, to appear in Phys. Rev. A, quant-ph/9708008.
- [24] P. Alsing and D. Cardimona, Phys. Rev. A, **45**, 1793 (1992).
- [25] To appear.
- [26] B. Yurke, S. L. McCall and J. R. Klauder, Phys. Rev. A, **33**, 4033 (1986); C. Brif and A. Mann, Phys. Rev. A, **54**, 4505 (1996); C. Brif and Y. Ben-Aryeh, Quant. Semiclass. Opt., **8**, 1 (1996).
- [27] R. Karp, F. Mansouri and J. Rno, to appear in Jour. Math. Phys., hep-th/9910173.
- [28] D. Vitali and P. Tombesi, Phys. Rev. A, **59**, 4178 (1999).
- [29] L. Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

## APPENDIX A: QUANTUM HOLONOMIES

In this appendix we shall give a mathematical review of the basic formalism concerning quantum geometric phases and their non-abelian generalizations. Although the material discussed below is by now standard (see for example Ref. [16] on which our presentation strongly relies) it has been here introduced for the sake of self-completeness of the paper and for making it more accessible to readers from the field of Quantum Information. For the necessary (elementary) background of fiber bundles theory we refer the reader to the book [15].

### 1. Abelian phases

In quantum theory the physical states are represented by *rays* in a separable Hilbert space  $\mathcal{H}$ . Mathematically this means that (pure) states are in a one-to-one correspondence with the elements of the *projective* space  $\mathbf{P}(\mathcal{H})$ . The latter is defined as the quotient space of  $\mathcal{H}$  with respect to the equivalence relation  $x \sim y \leftrightarrow \exists \lambda \in \mathbb{C} - \{0\} / y = \lambda x, \forall x, y \in \mathcal{H}$ . Alternatively one can consider the unit sphere in  $\mathcal{H}$ ,  $S^\infty := \{|\psi\rangle / \|\psi\| = 1\}$ , factored by the  $U(1)$  action  $(|\psi\rangle, e^{i\theta}) \mapsto e^{i\theta} |\psi\rangle$ . In this case one writes  $\mathbf{P}(\mathcal{H}) = S^\infty / U(1)$ .

The projection

$$\pi: S^\infty \rightarrow \mathbf{P}(\mathcal{H}) \quad (\text{A1})$$

defines a  $U(1)$  principal bundle over the base space  $\mathbf{P}(\mathcal{H})$  with total space  $S^\infty$ . This fiber bundle is the natural setting in which abelian holonomies i.e., Berry phases, appear. The fiber over the point  $|\psi\rangle\langle\psi| \in \mathbf{P}(\mathcal{H})$  is given by  $F_{|\psi\rangle} := \{e^{i\theta} |\psi\rangle / \theta \in [0, 2\pi)\}$ .

Any other principal  $U(1)$  bundle can be written in terms of (A1) as follows. Suppose  $\Phi$  to be a (smooth) map from the parameter manifold  $\mathcal{M}$  in  $\mathbf{P}(\mathcal{H})$ , then one can construct the *pull-back* bundle  $\Phi^* S^\infty$ , with total space  $\bigcup_{\lambda \in \mathcal{M}} F_{\Phi(\lambda)}$  and projection  $\pi^\Phi: F_{\Phi(\lambda)} \rightarrow \lambda$ . A map  $\lambda \mapsto |\psi_\lambda\rangle \in F_{\Phi(\lambda)}$  is a *section* of the bundle  $\Phi^* S^\infty$ .

The Schrödinger equation  $i \partial/\partial t |\psi\rangle = H |\psi\rangle$  determines a temporal evolution  $t \mapsto |\psi(t)\rangle$  in the total space  $S^\infty$  and via  $\pi$  in  $\mathbf{P}(\mathcal{H})$ . Moving the other way round i.e., from an evolution in  $\mathbf{P}(\mathcal{H})$  and obtaining an evolution in  $S^\infty$ , requires the introduction of a new key ingredient: a *connection*.

The  $U(1)$  action over  $S^\infty$  defines the *vertical* direction at each point along the fiber. The connection  $u(1)$ -valued 1-form  $A$  allows to define the *horizontal* direction as well. Once this field of directions is given one can realize the horizontal *lift* of any curve in the base.

To build the connection let us start by observing that  $S^\infty$  inherits from  $\mathcal{H}$  a natural hermitian structure. Such structure provides a natural mean for defining the horizontal directions at each point: the ones orthogonal to the fiber i.e., to  $|\psi\rangle$ . Given the curve  $t \mapsto |\psi(t)\rangle \in S^\infty$  we decompose the tangent vector  $|\dot{\psi}\rangle := d|\psi(t)\rangle/dt$  as follows  $|\dot{\psi}\rangle = \langle\psi|\dot{\psi}\rangle |\psi\rangle + |h_\psi\rangle$ . Where the horizontal component  $|h_\psi\rangle$  satisfies the relation  $\langle\psi|h_\psi\rangle = 0$ .

The connection can be evaluated explicitly by splitting the operators  $d/dt$  according its vertical and horizontal components:  $d/dt = \alpha \partial/\partial\theta + \sum_\mu B^\mu D_\mu$ , where  $\mu$  label the local chart coordinates  $(\lambda_\mu)$  of the base manifold  $\mathcal{M}$ . The horizontal operators  $D_\mu$  are referred to as the *covariant* derivatives, they are given by  $D_\mu := \partial/\partial\lambda_\mu + A_\mu \partial/\partial\theta$ . The  $u(1)$ -valued 1-form  $A := \sum_\mu A_\mu d\lambda_\mu$  is the connection form. Applying  $d/dt$  to  $|\psi(t)\rangle$  from the horizontal part one gets  $\langle\psi| D_\mu |\psi\rangle = \langle\psi| \partial/\partial\lambda_\mu + A_\mu \partial/\partial\theta |\psi\rangle = 0$ , from which  $A_\mu \langle\psi| \partial/\partial\theta |\psi\rangle = -\langle\psi| \partial/\partial\lambda_\mu |\psi\rangle$ . Since  $\partial/\partial\theta |\psi\rangle = i |\psi\rangle$  one obtains

$$A_\mu = i \langle\psi| \frac{\partial}{\partial\lambda_\mu} |\psi\rangle. \quad (\text{A2})$$

A section  $s: \mathcal{M} \mapsto S^\infty / \lambda \mapsto |\psi\rangle_\lambda$  of (A1), under a local coordinate change, transforms as  $|\psi\rangle_\lambda \mapsto \exp(i\theta(\lambda)) |\psi\rangle_\lambda$  (gauge transformation). From this relation and eq. (A2) one can check that the connection form changes according to  $A_\mu \mapsto A_\mu - \partial\theta/\partial\lambda_\mu$ .

Let us consider now a loop  $\gamma: [0, T] \mapsto \mathbf{P}(\mathcal{H})$  in the base space. Using the connection form (A2) we can lift it to the total space  $S^\infty$ , let  $|\tilde{\psi}(t)\rangle$  be such an horizontal lift. One can write  $|\tilde{\psi}(t)\rangle = e^{i f(t)} |\psi(t)\rangle$ , where  $|\psi(t)\rangle = (s \circ \gamma)(t)$ , is a *closed* in  $S^\infty$  obtained composing  $\gamma$  with a section. One has  $|\tilde{\psi}(T)\rangle = e^{i\beta} |\tilde{\psi}(0)\rangle$ , where  $\beta := f(T) - f(0)$ . From the above relations and the horizontality condition  $\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle = 0$ , one gets  $\beta = \int_0^T dt \langle\psi(t)|\dot{\psi}(t)\rangle$ . Since  $|\dot{\psi}\rangle = (\partial/\partial t + \sum_\mu \dot{\lambda}_\mu \partial/\partial\lambda_\mu) |\psi\rangle$ , and  $\theta(T) = \theta(0) \bmod 2\pi$ , by using Eq. (A2) one finds

$$\beta = \sum_{\mu} \int_{\gamma} d\lambda_{\mu} A_{\mu} = \int_{\gamma} A. \quad (\text{A3})$$

This expression of the  $U(1)$ -holonomy is manifestly gauge invariant and it does not depend on the Hamiltonian: it is of a pure *geometrical* origin.

In general the state vector is *not* horizontal  $\langle \psi | \dot{\psi} \rangle = -i \langle \psi | H(t) | \psi \rangle \neq 0$ , in this case besides the geometrical term (A3) one has a *dynamical* contribution to the phase given by  $\alpha := -\int_0^T \langle \psi(t) | H(t) | \psi(t) \rangle$ . At variance with the former this latter term depends on the Hamiltonian, moreover one can get rid of it by a gauge transformation  $|\psi(t)\rangle \mapsto U(t) |\psi(t)\rangle$ , where

$$U(t) := \exp[i \int_0^t d\tau \langle \psi(\tau) | H(\tau) | \psi(\tau) \rangle].$$

By introducing the *curvature* 2-form  $F := dA = \sum_{\mu\nu} F_{\mu\nu} d\lambda_{\mu} \wedge d\lambda_{\nu}$ ,  $F_{\mu\nu} = \partial A_{\mu} / \partial \lambda_{\nu} - \partial A_{\nu} / \partial \lambda_{\mu}$ , and using Stokes theorem, eq. (A3) can be rewritten as  $\beta := \int_{\Sigma} F$  where  $\Sigma$  is any surface having  $\gamma([0, T])$  as a boundary. A non-vanishing  $F$  describes a non-trivial geometry of the bundle (A1). One can generalize this construction to the more complex case in which the fiber is a  $n$ -dimensional complex space over which acts the group  $U(n)$ . This *non-Abelian* situation will be now reviewed.

## 2. Non Abelian generalization

Let us consider a family of Hamiltonians  $\{H(\lambda)\}_{\lambda \in \mathcal{M}}$  where  $\mathcal{M}$  is the control manifold. We shall consider loops  $\gamma$  in  $\mathcal{M}$ , and define  $H(t) := H_{\gamma(t)}$ ,  $t \in [0, T]$ . In general  $H(\lambda) = H_{\gamma(t)}$  has  $R$  different eigenvalues  $\{\varepsilon_l\}_{l=1}^R$  with degeneracies  $\{n_l(\lambda)\}$ . We shall assume that *no* level crossing occur i.e.,  $n_l(\lambda) = n_l$ . Let  $\Pi_l(\lambda)$  denote the projector over the eigen-space  $\mathcal{H}_l(\lambda) := \text{span}\{|\psi_l^{\alpha}(\lambda)\rangle\}_{\alpha=1}^{n_l}$ , of  $H(\lambda)$ . The spectral  $\lambda$ -dependent resolution of the Hamiltonians is then  $H(\lambda) = \sum_{l=1}^R \varepsilon_l(\lambda) \Pi_l(\lambda)$ . The mapping  $\lambda \mapsto |\psi_l^{\alpha}(\lambda)\rangle$  defines a section of the bundle

$$\frac{U(N)}{U(N - n_i)} \rightarrow \frac{U(N)}{U(n_i) \times U(N - n_i)} \quad (\text{A4})$$

where  $N := \dim \mathcal{H}$ . The total (base) space of the  $U(n_l)$ -principal bundle (A4) is known as the Stiefel (Grassmann) manifold and it is denoted by  $V_{N, n_l}(G_{N, n_l})$ . In the very same way as for the abelian case discussed above the hermitian structure over  $\mathcal{H}$  provides a natural notion of horizontality: tangent vectors are horizontal if they

are orthogonal to the fiber. Notice that one recovers the abelian ( $\mathcal{H} \cong \mathbb{C}^N$ ) case by setting  $n_l = 1$ , indeed  $G_{N, 1} = \mathbf{CP}^{N-1} = \mathbf{P}(\mathcal{H})$ .

The state vector evolves according the time-dependent Schrödinger equation  $i \partial_t |\psi(t)\rangle = H_{\gamma(t)} |\psi(t)\rangle$ . In the case in which the loop  $\gamma$ 's are traveled sufficiently slow one avoids transitions among different energy levels. In this *adiabatic* limit any initial preparation  $|\psi_0\rangle \in \mathcal{H}$  belonging to some energy eigen-space  $\mathcal{H}_l$  will be mapped, after the period  $T$ , onto:  $|\psi(T)\rangle = U_{\gamma} |\psi_0\rangle \in \mathcal{H}_l$ . For the sake of concreteness let us focus on the eigen-space associated with the eigenvalue  $\varepsilon_l = 0$ . Also let  $\{|\psi_{\alpha}(t)\rangle\}$  be the corresponding orthonormal basis at the instant  $t$ .

Let  $\eta_{\alpha}(t) = \sum_{\beta} U_{\alpha\beta}(t) \psi_{\beta}(t)$  be the solution of the Schrödinger equation with initial condition  $\eta_{\alpha}(0) = \psi_{\alpha}$ . By imposing, at each instant, the orthogonality i.e., horizontality, conditions  $\langle \eta_{\beta} | \eta_{\alpha} \rangle = \delta_{\alpha\beta}$  one gets by differentiation

$$\begin{aligned} 0 &= \langle \eta_{\beta} | \dot{\eta}_{\alpha} \rangle = \sum_{\delta} (\dot{U}_{\alpha\delta} \langle \eta_{\beta} | \psi_{\delta} \rangle + U_{\alpha\delta} \langle \eta_{\beta} | \dot{\psi}_{\delta} \rangle) \\ &= \dot{U}_{\alpha\delta} U_{\beta\delta}^* + \sum_{\delta\tau} U_{\alpha\delta} U_{\beta\tau}^* \langle \psi_{\tau} | \dot{\psi}_{\delta} \rangle, \end{aligned} \quad (\text{A5})$$

from which follows  $(U^{-1} \dot{U})_{\beta\alpha} = A_{\alpha\beta}$  where we have defined  $A_{\alpha\beta} := \langle \psi_{\beta} | \dot{\psi}_{\alpha} \rangle$ . From the above relations it follows that

$$U(t) = \mathbf{T} \exp \int_0^T d\tau A(\tau) = \mathbf{P} \exp \int_{\gamma} A, \quad (\text{A6})$$

where the  $u(n)$ -valued 1-form  $A = \sum_{\mu} A_{\mu} d\lambda_{\mu}$  is given by  $(A_{\mu})_{\alpha\beta} = \langle \psi_{\alpha} | \partial / \partial \lambda_{\mu} | \psi_{\beta} \rangle$ . Note that the matrix character of the connection  $A$  demands for the path ordering  $\mathbf{P}$  to take place. For a reparametrization of the loop  $\gamma$  in terms of a variable  $x \in [a, b]$  and for  $\mathcal{A}(x) = A_{\mu} \frac{d\lambda_{\mu}}{dx}$  it is formally defined by

$$\Gamma_A(\gamma) \equiv \mathbf{P} e^{\int_a^b \mathcal{A}(x) dx} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{P} \left( \int_a^b \mathcal{A}(x) dx \right)^n$$

with

$$\begin{aligned} &\frac{1}{n!} \mathbf{P} \left( \int_a^b \mathcal{A}(x) dx \right)^n := \\ &\int_a^b dx_1 \int_a^{x_1} dx_2 \dots \int_a^{x_{n-1}} dx_n \mathcal{A}(x_1) \mathcal{A}(x_2) \dots \mathcal{A}(x_n) \end{aligned} \quad (\text{A7})$$

where  $x_k = a + \frac{k}{n}(b - a)$ .